Orthogonally Decomposable Tensors

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Symmetric Tensors

T is an $\underbrace{n \times ... \times n}_{d \text{ times}}$ *symmetric tensor* with elements in a field $\mathbb{K}(= \mathbb{R}, \mathbb{C})$ if

$$T_{i_1 i_2 \dots i_d} = T_{i_{\sigma_1} i_{\sigma_2} \dots i_{\sigma_d}}$$

for all permutations σ of $\{1, 2, ..., d\}$. Notation: $T \in S^d(\mathbb{K}^n)$.

Example ($d = 2$)

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{22} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix}$$

Example ($n = 3, d = 3$)

$$T = \underbrace{\begin{pmatrix} T_{111} & T_{112} & T_{113} \\ T_{112} & T_{122} & T_{123} \\ T_{113} & T_{123} & T_{133} \end{pmatrix}}_{T_{1\cdot\cdot}}, \underbrace{\begin{pmatrix} T_{112} & T_{122} & T_{123} \\ T_{122} & T_{222} & T_{223} \\ T_{123} & T_{223} & T_{233} \end{pmatrix}}_{T_{2\cdot\cdot}}, \underbrace{\begin{pmatrix} T_{113} & T_{123} & T_{133} \\ T_{123} & T_{223} & T_{233} \\ T_{133} & T_{233} & T_{333} \end{pmatrix}}_{T_{3\cdot\cdot}}.$$

Symmetric Tensors and Polynomials

An equivalent way of representing a symmetric tensor $T \in S^d(\mathbb{K}^n)$ is by a *homogeneous polynomial* $f \in \mathbb{K}[x_1, ..., x_n]$ of degree d.

Example ($d = 2$)

In the case of matrices,

$$\begin{aligned} f(x_1, ..., x_n) &= x^T T x \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{13} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \sum_{i,j} T_{ij} x_i x_j. \end{aligned}$$

Symmetric Tensors and Polynomials

For general $T \in S^d(\mathbb{K}^n)$,

$$f(x_1, ..., x_n) = T \cdot x^d := \sum_{i_1,\dots,i_d=1}^{n} T_{i_1 \dots i_d} x_{i_1} \dots x_{i_d}$$

$$= \sum_{j_1+\cdots+j_n=d} \binom{d}{j_1, \dots, j_n} T_{\underbrace{1\dots1}_{j_1}\dots\underbrace{n\dots n}_{j_n}} x_1^{j_1} \dots x_n^{j_n}$$

$$= \sum_{j_1+\cdots+j_n=d} u_{j_1,\dots,j_n} x_1^{j_1} \dots x_n^{j_n}.$$

Example $(n = 3, d = 2)$

For 3×3 matrices,

$$f(x_1, x_2, x_3) = \sum_{i_1,i_2=1}^{3} T_{i_1 i_2} x_{i_1} x_{i_2}$$

$$= \underbrace{T_{11}}_{u_{2,0,0}} x_1^2 + \underbrace{2T_{12}}_{u_{1,1,0}} x_1 x_2 + \underbrace{2T_{13}}_{u_{1,0,1}} x_1 x_3 + \underbrace{T_{22}}_{u_{0,2,0}} x_2^2 + \underbrace{2T_{23}}_{u_{0,1,1}} x_2 x_3 + \underbrace{T_{33}}_{u_{0,0,2}} x_3^2.$$

Symmetric Tensor Decomposition

For a tensor $T \in S^d(\mathbb{K}^n)$, a decomposition has the form

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d}.$$

If $f \in \mathbb{K}[x_1, ..., x_n]$ is the corresponding polynomial, then

$$f(x_1, ..., x_n) = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d = \sum_{i=1}^{r} \lambda_i (v_{i1}x_1 + v_{i2}x_2 + \cdots + v_{in}x_n)^d.$$

The smallest r for which such a decomposition exists is the *symmetric rank* of T.

Facts about Symmetric Tensor Decomposition

- The rank depends on the field $\mathbb{K}$.

Example ($d = 3, n = 2$)

Consider

$$A = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \in S^3(\mathbb{R}^2).$$

A has symmetric rank 3 over $\mathbb{R}$:

$$A = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}^{\otimes 3} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}^{\otimes 3} - 2\begin{bmatrix}1\\0\end{bmatrix}^{\otimes 3},$$

whereas it has symmetric rank 2 over $\mathbb{C}$:

$$A = \frac{i}{2}\begin{bmatrix}-i\\1\end{bmatrix}^{\otimes 3} - \frac{i}{2}\begin{bmatrix}i\\1\end{bmatrix}^{\otimes 3}, \text{ where } i = \sqrt{-1}.$$

Facts about Symmetric Tensor Decomposition

- The rank strata: $\mathcal{Y}_r := \{T \in S^d(\mathbb{K}^n) : \text{rank}(T) \leq r\}$ are usually not closed.

Example (Matrices)

For matrices the rank strata ARE closed:

$$\mathcal{Y}_r = \{T \in S^2(\mathbb{K}^n) : \ \text{rank}(T) \leq r\}$$

$$= \text{zero set of } (r+1) \times (r+1) \text{ minors of } T,$$

e.g. when $n = 3$,

$$\mathcal{Y}_1 = \{T \in S^2(\mathbb{K}^3) : \ \text{rank}(T) \leq 1\}$$

$$= \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} : \begin{matrix} x_{11}x_{22} - x_{12}^2 = 0, x_{11}x_{23} - x_{13}x_{12} = 0, \\ x_{11}x_{33} - x_{13}^2 = 0, x_{12}x_{23} - x_{13}x_{22} = 0 \\ x_{12}x_{33} - x_{13}x_{23} = 0, x_{22}x_{33} - x_{23}^2 = 0 \end{matrix} \right\},$$

which is a closed set.

Facts about Symmetric Tensor Decomposition

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Example

Let $\epsilon \neq 0$ and x, y non-collinear vectors.

$$A_\epsilon = \epsilon^2(x + \epsilon^{-1}y)^{\otimes 3} + \epsilon^2(x - \epsilon^{-1}y)^{\otimes 3}.$$

When $\epsilon \to 0$,

$$A_\epsilon \to A_0 = 2(x \otimes y \otimes y + y \otimes x \otimes y + y \otimes y \otimes x),$$

which has symmetric rank 3:

$$A_0 = (x + y)^{\otimes 3} - (x - y)^{\otimes 3} - 2y^{\otimes 3}.$$

Facts about Symmetric Tensor Decomposition

- The *generic rank* of tensors in $S^d(\mathbb{C}^n)$, denoted by $\overline{R}_S(d,n)$ is the smallest r such that "almost all" $T \in S^d(\mathbb{C}^n)$ have symmetric rank at most r.

Example ($d = 2$)

The generic rank of $n \times n$ marices is n.

Example ($n = 3, d = 3$)

The generic rank for tensors $T \in S^3(\mathbb{C}^3)$ is $\overline{R}_S(3,3) = 4$. All tensors $T \in S^3(\mathbb{C}^3)$ that have symmetric rank at most 3 satisfy one polynomial equation $f(T) = 0$, called the *Aronhold invariant*.

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Theorem (Alexander-Hirschowitz)

For $d > 2$,

$$\overline{R}_S(d,n) = \left\lceil \frac{1}{n}\binom{n+d-1}{d} \right\rceil$$

except for the cases: $(d,n) \in \{(3,5),(4,3),(4,4),(4,5)\}$*, where it should be increased by* 1*.*

Facts about Symmetric Tensor Decomposition

- When is the symmetric tensor decomposition unique?

Theorem

For all $r < \overline{R}_S(d,n)$, the general element of rank r in $S^d(\mathbb{C}^n)$ has a unique (up to scaling) decomposition $T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d}$ with the only exceptions

(1) $(d,n) \in \{(3,5),(4,3),(4,4),(4,5)\}$, *where there are infinitely many decompositions,*

(2) *rank 9 in $S^6(\mathbb{C}^3)$, where there are two decompositions,*

(3) *rank 8 in $S^4(\mathbb{C}^4)$, where there are two decompositions.*

Orthogonal Tensor Decomposition

An *orthogonal decomposition* of a symmetric tensor $T \in S^d(\mathbb{K}^n)$ is a decomposition

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d} \quad \text{with corresponding} \quad f = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d$$

such that the vectors $v_1, \dots, v_r$ are orthonormal. In particular, $r \leq n$.

Definition

A tensor $T \in S^d(\mathbb{K}^n)$ with corresponding f is *orthogonally decomposable*, for short *odeco*, if it has an orthogonal decomposition.

Examples

1. All symmetric matrices are odeco: by the spectral theorem

$$T = V^T \Lambda V = \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes 2},$$

where $v_1, \ldots, v_n$ is an orthonormal basis of eigenvectors.

2. The Fermat polynomial: If $v_i = e_i$, for $i = 1, \ldots, n$, then

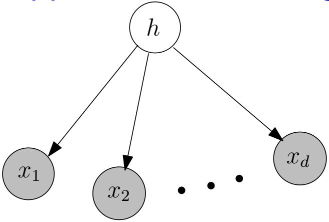
$$f(x_1, \ldots, x_n) = x_1^d + x_2^d + \cdots + x_n^d,$$

$$T = e_1^{\otimes d} + e_2^{\otimes d} + \cdots + e_n^{\otimes d}.$$

3. If $V = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 1 & -1 \\ 4 & 1 & -7 \end{pmatrix}$, then

$$f(x, y, z) = (x + 3y + z)^3 + (-2x + y - z)^3 + (4x + y - 7z)^3.$$

An Application: Exchangeable Single Topic Models



Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are *i.i.d* random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

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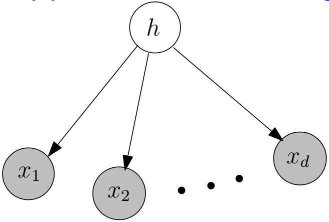

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Then, the joint distribution of $x_1, ..., x_d$ is an $\underbrace{n \times n \times \cdots \times n}_{d \text{ times}}$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j) \prod_{i=1}^{d} \mathbb{P}(x_i | h = j) = \sum_{j=1}^{k} w_j \mu_j^{\otimes d}.$$

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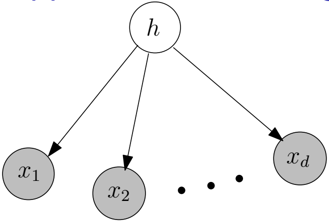

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Given T, to recover the parameters w, μ, use a transformation $T \mapsto T_{od}$ and decompose T_{od}.

Eigenvectors of Tensors

Consider a symmetric tensor $T \in S^d(\mathbb{K}^n)$.

Example ($d = 2$)

T is an $n \times n$ matrix and $w \in \mathbb{K}^n$ is an eigenvector if

$$Tw = \begin{pmatrix} \vdots \\ \sum_{j=1}^n T_{i,j} w_j \\ \vdots \end{pmatrix} = \lambda w.$$

Example ($d = 3$)

T is an $n \times n \times n$ tensor and $w \in \mathbb{K}^n$ is an eigenvector if

$$Tw^2 := \begin{pmatrix} \vdots \\ \sum_{j,k=1}^n T_{i,j,k} w_j w_k \\ \vdots \end{pmatrix} = \lambda w.$$

Eigenvectors of Tensors

Definition

- Given a symmetric tensor $T \in S^d(\mathbb{K}^n)$, an *eigenvector* of T with *eigenvalue* λ is a vector $w \in \mathbb{K}^n$ such that

$$Tw^{d-1} := \begin{pmatrix} \vdots \\ \sum_{i_2,\ldots,i_d=1}^{n} T_{i,i_2,\ldots,i_d} w_{i_2} \ldots w_{i_d} \\ \vdots \end{pmatrix} = \lambda w.$$

  Two eigenvector-eigenvalue pairs (w, λ) and (w', λ') are equivalent if there exists $t \in \mathbb{K} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

- For the corresponding $f \in \mathbb{K}[x_1, \ldots, x_n]$, $w \in \mathbb{K}^n$ is an *eigenvector* with *eigenvalue* λ if

$$\nabla f(w) = d\lambda w.$$

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$$\nabla f(w) = d\lambda w.$$

Therefore, the eigenvectors of f are given by the vanishing of the

$$2 \times 2 \text{ minors of the matrix } [\nabla f(x)|x].$$

Eigenvectors of Tensors

Example

Let

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} \text{ and } f(x, y, z) = x^3 + y^3 + z^3.$$

Then, $(x, y, z)^T$ is an eigenvector of f if and only if the 2×2 minors of the matrix $\begin{bmatrix} & x \\ \nabla f & y \\ & z \end{bmatrix} = \begin{bmatrix} 3x^2 & x \\ 3y^2 & y \\ 3z^2 & z \end{bmatrix}$ vanish. Therefore,

$$x^2y - xy^2 = x^2z - xz^2 = y^2z - yz^2 = 0.$$

This is equivalent to

$$xy(x - y) = xz(x - z) = yz(y - z) = 0.$$

The solutions are (up to scaling):

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.$$

Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ is an odeco tensor, i.e. $v_1, \dots, v_n$ are orthonormal, then the vectors v_k, $k = 1, \dots, n$ are eigenvectors of T with corresponding eigenvalues λ_k, $k = 1, \dots, n$:

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_k (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

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- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?

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- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?
- Are these all of the eigenvectors of an odeco tensor?

Robust Eigenvectors

Definition

A unit vector $u \in \mathbb{R}^n$ is a *robust eigenvector* of a tensor $T \in S^d(\mathbb{R}^n)$ if there exists $\epsilon > 0$ such that for all $\theta \in \mathcal{B}_\epsilon(u) = \{u' : ||u - u'|| < \epsilon\}$, repeated iteration of the map

$$\theta \mapsto \frac{T\theta^{d-1}}{||T\theta^{d-1}||}, \tag{1}$$

starting from θ converges to u.

Theorem (Anandkumar et al.)

Let $d = 3$ and let T have an orthogonal decomposition $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ as in the definition.

1. *The set of $\theta \in \mathbb{R}^n$ which do not converge to some v_i under repeated iteration of (1) has measure 0.*
2. *The set of robust eigenvectors of T is equal to $\{v_1, v_2, ..., v_k\}$.*

The Tensor Power Method

The tensor power method consists of repeated iteration of the map

$$u \mapsto \frac{Tu^{d-1}}{||Tu^{d-1}||},$$

or equivalently,

$$u \mapsto \frac{\nabla f(u)}{||\nabla f(u)||}.$$

Algorithm

Input: An odeco tensor T.
Output: An orthogonal representation of T.

Repeat
    Find $v_i \leftarrow$ power method output starting from a random $u \in \mathbb{R}^n$.
    Recover $\lambda_i = T \cdot v_i^d$.
    $T \leftarrow T - \lambda_i v_i^{\otimes d}$.
Return $v_1, ..., v_n$ and $\lambda_1, ..., \lambda_n$.

The Number of Eigenvectors of a Tensor

<u>Recall</u>: Given a tensor $T \in S^d(\mathbb{C}^n)$ with corresponding polynomial f, the eigenvectors $x \in \mathbb{C}^n$ are the solutions to the equations given by the 2×2 minors of the matrix

$$\left[\nabla f(x) | x\right].$$

Theorem (Sturmfels and Cartwright)

If a tensor $T \in S^d(\mathbb{C}^n)$ has finitely many eigenvectors, then their number is $\frac{(d-1)^n-1}{d-2}$.

Eigenvectors of Odeco Tensors

Example

Let $m = n = 3$ and consider the matrix with orthogonal rows $V = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 1 & -1 \\ 4 & 1 & -7 \end{pmatrix}$.

$$f(x, y, z) = (x + 3y + z)^3 + (-2x + y - z)^3 + (4x + y - 7z)^3$$

$$= 57x^3 + 69x^2y + 33xy^2 + 29y^3 - 345x^2z - 138xyz + 3y^2z + 585xz^2 + 159yz^2 - 343z^3.$$

Then, the eigenvectors satisfy the equations given by the 2×2 minors of

$$\begin{bmatrix} 171x^2 + 138xy + 33y^2 - 690xz - 138yz + 585z^2 & x \\ 69x^2 + 66xy + 87y^2 - 138xz + 6yz + 159z^2 & y \\ -345x^2 - 138xy + 3y^2 + 1170xz + 318yz - 1029z^2 & z \end{bmatrix}.$$

Let $I \subseteq \mathbb{C}[x_1, ..., x_n]$ be the *ideal* generated by the 2×2 minors of the matrix $\big[\nabla f(x) \big| x\big]$.
Then, $\mathcal{V}(I)$ is the *variety* of eigenvectors of f.

Eigenvectors of Odeco Tensors

Example (...continued)

We use the computer algebra software Macaulay2 to decompose the ideal I:

$$
\begin{aligned}
I = & \langle -y + 3z, x - z\rangle \bigcap \langle y + z, x - 2z\rangle \bigcap \langle 7y + z, 7x + 4z\rangle \\
& \bigcap \langle 64y + 61z, 64x - 119z\rangle \bigcap \langle -29y + 109z, 29x - 40z\rangle \\
& \bigcap \langle 2y + 5z, 46x - 101z\rangle \bigcap \langle 85y + 229z, 85x - 206z\rangle.
\end{aligned}
$$

In other words, the set of eigenvectors is the union of the solutions to the much simpler systems of equations above.

Eigenvectors of Odeco Tensors

Odeco tensors are nice because we can characterize all of their eigenvectors.

Theorem

Let $f \in S^d(\mathbb{C}^n)$ be an odeco tensor with $f(x_1, ..., x_n) = \sum_{i=1}^{n} \lambda_i (Vx)_i^d$, where V is an orthogonal matrix. Then, f has $\frac{(d-1)^n - 1}{d-2}$ eigenvectors, which represent all of the fixed points of the gradient map in projective space $\mathbb{CP}^n$. Explicitly, the eigenvectors are

$$(x_1 : \cdots : x_n) =$$
$$= V^T(\eta_1 \lambda_1^{-\frac{1}{d-2}} : ... : \eta_{k-1} \lambda_{k-1}^{-\frac{1}{d-2}} : \lambda_k^{-\frac{1}{d-2}} : 0 : ... : 0)^T,$$

where $k = 1, ..., n$ and $\eta_1, ..., \eta_{k-1}$ are $d - 2^{nd}$ roots of unity.

Eigenvectors of Odeco Tensors

Example ($d = 3, n = 3$)

Let

$$f(x, y, z) = x^3 + y^3 + z^3.$$

Then, $V = I$, the identity matrix and the eigenvectors of f are:

$k = 1$ $(1 : 0 : 0)^T, (0 : 1 : 0)^T, (0 : 0 : 1)^T$

$k = 2$ $(1 : 1 : 0)^T, (1 : 0 : 1)^T, (0 : 1 : 1)^T$

$k = 3$ $(1 : 1 : 1)^T$.

Eigenvectors of Odeco Tensors

Example (n=3, d=3)

$$f(x,y,z) = (x+3y+z)^3 + (-2x+y-z)^3 + (4x+y-7z)^3$$

$$= \sqrt{11}^3\left(\frac{1}{\sqrt{11}}(x+3y+z)\right)^3 + \sqrt{6}^3\left(\frac{1}{\sqrt{6}}(-2x+y-z)\right)^3 + \sqrt{66}^3\left(\frac{1}{\sqrt{66}}(4x+y-7z)\right)^3,$$

and $\lambda_1 = \sqrt{11}^3, \lambda_2 = \sqrt{6}^3, \lambda_3 = \sqrt{66}^3$, $V = \begin{pmatrix} \frac{1}{\sqrt{11}}(1,3,1) \\ \frac{1}{\sqrt{6}}(-2,1,-1) \\ \frac{1}{\sqrt{66}}(4,1,-7) \end{pmatrix}$. Then, the eigenvectors are

$k=1$ $V^T(\lambda_1^{-1}:0:0) = (1:3:1)$, $V^T(0:\lambda_2^{-1}:0) = (-2:1:0)$, $V^T(0:0:\lambda_3^{-1}) = (4:1:-7)$.

$k=2$ $V^T(\lambda_1^{-1}:\lambda_2^{-1}:0) = (206:-229:85)$, $V^T(\lambda_1^{-1}:0:\lambda_3^{-1}) = (40:109:29)$, $V^T(0:\lambda_2^{-1}:\lambda_3^{-1}) = (119:-61:64)$.

$k=3$ $V^T(\lambda_1^{-1}:\lambda_2^{-1}:\lambda_3^{-1}) = (101:-230:46)$.

The Set of Odeco Tensors

- Parametric representation:
  The set of orthogonally decomposable tensors can be parametrized by $\mathbb{K}^n \times O_n(\mathbb{K})$:
  $$\lambda, V \mapsto \sum_{i=1}^{n} \lambda_i (v_i \cdot x)^n.$$
- Implicit representation:
  The set of orthogonally decomposable tensors can also be represented as the solutions to a set of equations.

Definition

The *odeco variety* is the Zariski closure of the set of all odeco tensors in $S^d(\mathbb{C}^n)$.

Goal: find equations defining this variety.

The Odeco Variety

Let $T \in S^d(\mathbb{C}^n)$. Let $\mathcal{F}$ be the set of the following equations:

fix $i_1, ..., i_{d-3} \in \{1, 2, ..., n\}$,

for each $i < j, k < l \in \{1, 2, ..., n\}$, consider the equation

$$p_{i_1,\ldots,i_{d-3},i,j,k,l} :=$$
$$= \sum_{s=1}^{n} T_{i_1,\ldots,i_{d-3},i,j,s} T_{i_1,\ldots,i_{d-3},k,l,s} - T_{i_1,\ldots,i_{d-3},i,l,s} T_{i_1,\ldots,i_{d-3},k,j,s}.$$

Lemma

The equations $\mathcal{F}$ vanish on the set of orthogonally decomposable tensors.

The Odeco Variety

Conjecture

The odeco variety is given by $\mathcal{V}(\mathcal{F})$.

Thank you!

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